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# Talking About Rates Conceptually, Part I: A Teacher's Struggle [1]

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Classroom discourse and communication are issues central to current reform in mathematics education. For reform to happen in classrooms, teachers will have to teach from a conceptual curriculum. To do so they must be sensitive to children's thinking during instruction and shape their instructional actions accordingly—to ensure that children hear what they intend them to hear. The article examines how one teacher's way of knowing mathematics was reflected in the language he used in teaching concepts of rate to one student during a teaching experiment. The teacher's conceptualizations of rate, although strong and elaborate, were encapsulated in the language of numbers and operations, and this undermined his effort to help the student understand rates conceptually. We examine the teacher's language in light of his intentions and the student's interpretations of the tasks. We discuss implications for instruction that is based on a conceptual curriculum and is sensitive and responsive to children's thinking.

Interest among mathematics education researchers in the study of communication and language in the mathematics classroom, in patterns of classroom discourse, and in establishing norms for discourse has been on the rise in recent years (Cobb, Yackel, & Wood, 1989; Lampert, 1990; Lo, Wheatley, & Smith, 1991). These studies have contributed to our increased appreciation of the role of communication in teaching and learning mathematics. For the most part, they have focused on how social norms of classroom interaction that facilitate the construction of mathematical knowledge are established and maintained. They have looked at discourse and communication as interaction among individuals, but have tended to concentrate on the interactions per se and less so on the meanings, intentions, and interpretations made by the individuals involved in discourse at the moments of conversing. That is, they have not examined students' and teacher's thinking during those social interactions wherein mathematical meanings are constituted and through which norms come to be established. One exception to this is Bauersfeld (1980), in which he examined a teacher's intentions, a student's understandings of the teacher's intentions, and how these interplayed during conversations surrounding an assignment on isometric transformations of the plane.

Although there is no shortage of theoretical frameworks for the study of communication

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in general (Hockett, 1960; von Glasersfeld, 1977), and in the classroom, in particular (Mehan, 1982; Wertsch & Toma, 1995), in this paper we use a simple-minded model of communication among people. It is that people express what they have in mind and interpret what they hear with varying degrees of reflectiveness and articulateness. Ongoing conversations are dialectically reflexive (Steier, 1989). Each party in a conversation is living within his or her own world of ideas, but takes into account the sense he or she makes of other people's expressions, and attributes, knowingly or unknowingly, intentions and motivations to others in the conversation (Bauersfeld, 1988; Cobb, 1990; Richards, 1991; Thompson, 1979). Conversations are quite like non-linear, chaotic systems, in that the possible directions they might take at any moment is a function of the participants' current understandings and intentions, and those understandings and intentions are influenced by directions taken within the conversation (MacKay, 1955; MacKay, 1964; MacKay, 1965; Mehan, 1982). Conversations unfold, they don't just happen. To quote Bauersfeld, "*The [conversation] is constituted at every moment through the interaction of reflective subjects*" (Bauersfeld, 1980).

Conversations "stay on track" or "go astray" (from participants' viewpoints) according to how easily (from an observer's viewpoint) participants can assimilate others' communicative actions to their current understandings of what the conversation is about. From the participants' viewpoints, conversations can seem to be "on track" when from an observer's viewpoint they have gone astray; each participant in the conversation may think he or she understands what the other is saying, yet, from an observer's view, their understandings are substantially different. This last case will be of primary interest in our analysis of obstacles to teaching concepts of rate. We will use occasions of rampant miscommunication between a teacher and a student as a context for discussing their understandings of rate, the language in which they express those understandings, and how those understandings lead eventually to dysfunctional instructional interactions (Thompson, Philipp, Thompson, & Boyd, 1994).

## COMMUNICATION AND CURRICULUM REFORM

Classroom discourse and communication are issues central to the current reform in mathematics education. For curricular reform to happen in classrooms, teachers must teach from the basis of a conceptual curriculum. That means they must be sensitive to children's thinking during instruction, and they must shape their instructional actions so that children actually hear what is intended. Various forms of this question can be asked: "How can we expect teachers to teach with sensitivity to the learning of 25-30 children?" We turned this question on its head, instead asking, "Suppose a teacher is relieved of all distractions and is responsible for attempting to influence one student's thinking so that the student ends up constructing a concept of speed. What might happen?" By investigating this question, we reduced the social complexity and demands of whole class interaction, thereby hoping to gain insight into the cognitive and attitudinal constraints teachers face when they attempt to influence children's thinking so that they construct mathematical concepts.

In this article and its sequel we discuss issues of communication between a teacher and a student as they talk about ideas of distance, time, and speed. We have separated our discussion into two articles. In the present article we examine the reflexive relationship between a teacher's and a student's "ways of knowing" the ideas of speed and rate and the mechanisms by which this relationship evolved over two days of instruction. In the sequel (Thompson & Thompson, in press) we focus more closely on issues pertaining to the teacher's understanding of rate in relation to matters of cognition, communication, and pedagogy.

## THE TEACHER

Bill teaches at Local Middle School, where he was hired to teach mathematics and science in grades 6-8. Before coming to Local Middle School, Bill taught high school physics, chemistry, and physical science for six years. At the time of this study, Bill was in his second year of teaching middle school and his assignment consisted of six mathematics classes: one sixth-grade, four seventh-grade, and one eighth-grade.

Bill received an undergraduate degree in veterinary medicine in 1965. After rising to the rank of vice president of a major biological company and establishing his own companies in 1976, he retired from private business in 1986 and subsequently became interested in education. He received teaching credentials for biological and physical sciences in 1988 and 1989, respectively and an "emergency credential" in mathematics. Bill’s college mathematics preparation consisted of two semesters of Calculus.

Bill was very adept at reasoning proportionally, whether relationships were direct or inverse. He meaningfully and creatively solved each proportional reasoning item on a test developed by the Rational Number Project (Post, Harel, Behr, & Lesh, 1991). Further evidence of his ability to reason in terms of direct and inverse proportions came from his solution to this problem: “A pan balance is off center. An object put on one side weighs 10 lb. The same object put on the other side weighs 40 lb. How much does the object weigh?” Bill saw easily that the two pans’ distances from the fulcrum had to be inversely proportional to the ratio of the weights. So, if $d_1/d_2$ is the ratio of the distances from the fulcrum, where $d_1$ is the shorter distance, and if we let $x$ stand for the object’s weight, then the ratios $x/40$ and $10/x$ must be equal to $d_1/d_2$, and therefore must be equal to each other.

## PREPARING FOR THE TEACHING EXPERIMENT

We arranged for a substitute teacher to take one of Bill’s classes each day for four days, April 29-May 2, 1991. Bill chose to work with Ann, one of the stronger students in her sixth grade class. Her CTBS scores the previous year were: computation 59%-ile; concepts and applications 82%-ile. Bill and Ann met 40 minutes each day in a quiet, isolated room.

The ideas of speed and average speed were the focus of instruction. The objective of focusing on speed and average speed was that Ann come to understand speed as a rate.

The materials for the teaching experiment was a set of questions having to do with a computer program called Over & Back (Thompson, 1990a). Over & Back presents two animals, Turtle and Rabbit, who run along a number line (Figure 1). Both can be assigned speeds at which to run. Turtle’s speed can be assigned two values: one for its speed while running “over”, the other for its speed “back.” Rabbit’s speed can be assigned only one value, which applies to both its trip over and its trip back. Each animal can be made to run separately from the other, or they can be made to run simultaneously (as in a race). A timer shows elapsed time as either of the animals runs. One can press the “Pause” button to interrupt a race; when “paused,” the distances traveled by either or both animals is displayed on the screen (Figure 2). Any assigned value can be changed during a pause; the animals will renew their race with speeds that correspond to the new values.

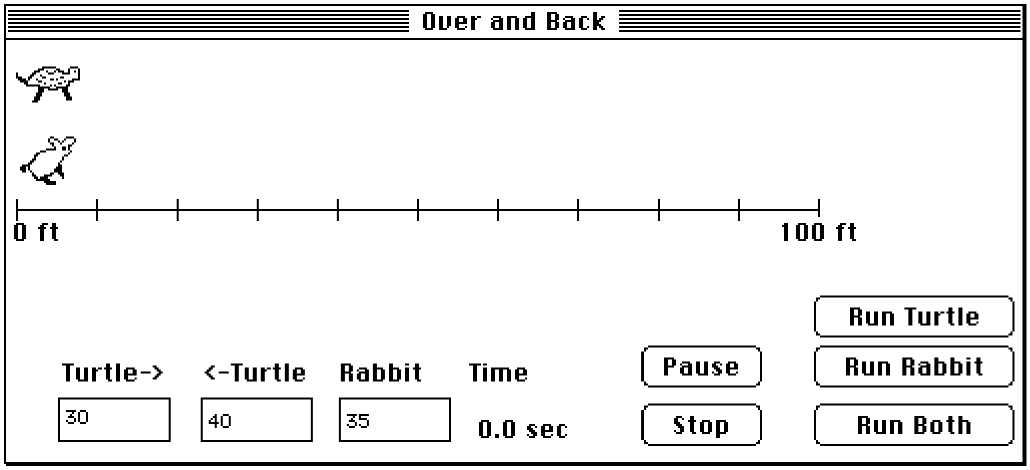


Figure 1. Over & Back's startup screen.

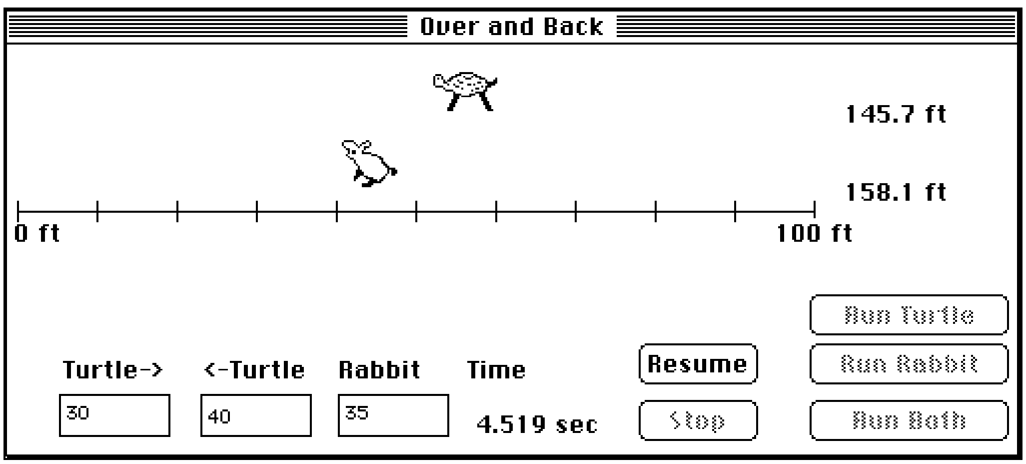


Figure 2. Turtle and Rabbit after pressing *Pause*.

A sample of each activity's questions is given in Table 1. The questions and the sequence in which they are given derives from previous teaching experiments on concepts of speed and rate (Thompson, 1994; Thompson & Thompson, 1992). The focus of these teaching experiments was to investigate students' construction of concepts of speed and the relationship between students' concepts of speed and rate.

| Activity | Example |
|---|---|
| 1 | Turtle is going over at 20 feet per second, coming back at 30 feet per second. How much time does he take? |
| 2 | Give Rabbit a speed that will make him go over and back in 7 seconds. |
| 3 | Turtle goes over at some speed and comes back at 70 feet per second. Rabbit goes over and back at 30 feet per second. Give Turtle a speed over so that he and Rabbit tie. |
| 4 | Sue paid \$9.46 for Yummy candy bars at \$0.43 per bar, and she paid \$6.08 for Zingy candy bars. Sue bought 38 of these candy bars. What was the price of a Zingy candy bar? |

Table 1. Examples of tasks used during the teaching experiment.

*A Conceptual Curriculum for Speed*

The image of speed we intended students construct through this unit is composed of these items, which themselves are constructions:

1. speed is a quantification of motion;
2. completed motion involves two completed quantities—distance traveled and amount of time required to travel that distance (this must be available to students both in retrospect and in anticipation);
3. speed as a quantification of completed motion is made by multiplicatively comparing distance traveled and amount of time required to go that distance;
4. there is a direct proportional relationship between distance traveled and amount of time required to travel that distance. That is, if you go $m$ distance units in $s$ time units at a constant speed, then at this speed you will go $a/b \cdot m$ distance units in $a/b \cdot s$ time units.

   Described imagistically, to say that there is a direct proportional relationship between distance and time means that one "sees" that partitioning a traveled total distance implies a proportional partition of total time required to travel that distance, and partitioning the total time required to travel a distance implies a proportional partition of the distance traveled.

In previous research that builds upon Piaget's (Piaget, 1970) theory of the epigenesis of speed we found that this image develops through a progressive internalization of measuring total distances in units of speed-lengths—distance traveled in one unit of time (Thompson, 1994; Thompson & Thompson, 1992). Children first internalize the process of measuring a total-distance-traveled in units of speed-length (the measurement producing an amount of time required to travel that distance). When children have internalized the measurement of a total distance in units of speed-length they can anticipate that traveling a distance at some constant speed will produce an amount of time. This implies that children first conceive speed as a distance and time as a ratio. With this anticipation they can reason about their image of completed motion, thinking about corresponding segmentations of accumulated distance and accumulated time. Their internalization of the dual measurement process provides a foundation for their conceptualizing constant speed as a rate. Prior to internalizing this measurement process, children are unable to conceptualize a situation where one is to determine a speed at which one will travel a given distance in a given amount of time. This is not to say that they are unable to answer questions about at what speed one must travel to go a given distance in a given amount of time. Rather, the only questions they can answer are those where they can use the equivalent of guess-and-test, looking for a distance (speed-length) which will produce a given travel time when they measure the given total distance in units of the speed-length.

*Bill's Preparation for the Experiment*

Bill had been working with us for six months in an ongoing seminar on instruction to support students' quantitative reasoning. In that time he met with one or both of us twice weekly to create a curriculum for his class, to discuss what we hoped students would learn, and to discuss principles of learning, teaching, and pedagogy as they pertained to current events surrounding Bill's teaching.

In addition to regular seminar meetings, Pat met with Bill on three occasions over two weeks to go over various issues related to Bill's upcoming experiment: (a) the mechanics of the program; (b) situations and questions that Bill would initiate with Ann; (c) objectives of the unit; (d) possible difficulties we expected Ann might have; and (e) pedagogical strategies Bill might take in relation to Ann's difficulties.

Pat did not address all five issues at once. First he worked with Bill as if he were a student (keeping in mind that Bill was a teacher, not a student) and as if Pat were teaching him about concepts of speed and rate (Thompson, 1994; Thompson & Thompson, 1992). The aim of this meeting was twofold: to familiarize Bill with Over & Back and the tasks he would present to Ann, and to provide Bill an occasion to experience the tasks in a way that might resemble Ann's experience. Bill had a computer in his classroom and a computer at home on which to work.

After Bill had gone through the activities as a student, Pat and he discussed difficulties that Ann might encounter during the teaching experiment. Pat stressed that Ann probably would experience insurmountable difficulty with Activity 2 (Table 1) unless two things happened: (a) during Activity 1 (Table 1) she came to anticipate that a completed trip would involve both traveling the total distance and accumulating a total amount of time; and (b) she came to see a proportional relationship between completed distance and completed time.

Pat emphasized that Bill should feel free to vary the tasks in Activity 1 if he sensed that Ann was thinking of speed strictly as a length and of time strictly as a ratio. One possible variation would be to use "ugly" numbers (e.g., fractions) instead of the "friendly" numbers that appeared in the activity sheets.

Other points emphasized were

- Don't let Ann become bogged down with doing calculations. Instead, orient her to writing her calculations as expressions (indicated operations), then let her use a calculator.
- Shape your language so that it is about the situation, not about numbers and arithmetic operations.
- Speak about the dynamics of the situations, not about static states.
- Try to get Ann to *anticipate* that, when thinking about traveling at some speed over

some distance, there automatically will be a total amount of time as well as a total amount of distance traveled.

- Use pedagogical principles that had been established over the preceding six months of seminars: When saying a number, say what it is a number of (and insist that Ann do the same); when stating an arithmetic operation, say what you are finding by doing that operation (and ask the same of Ann).

To give Bill an opportunity to internalize this advice, Pat pretended to be a student and responded to Bill's instruction as if he had only initial conceptions of speed-as-distance. In retrospect, Bill's reaction to Pat's role-playing is significant. Bill struggled with the idea that measuring the total distance in units of speed-length is not "just the same thing as division," as he maintained. He also appeared dubious that determining a speed for a fixed distance and fixed time was different, conceptually, from determining time for a fixed distance and speed, "since you divide in both cases." At one point Bill suggested, in good spirits, that Pat might be "overdoing it a little" in the difficulties he pretended to have. Bill allowed that Pat might have a point in drawing the distinctions he did, and that he (Bill) would think about it some more.

## THE TEACHING EXPERIMENT

In each session, both Bill and Ann sat in front of a large table. A computer running Over & Back was on the table, and there was ample space for Ann to write on scratch paper. Each session was videotaped and transcribed. In this article we report only the first two sessions. It was in these sessions that Ann's and Bill's difficulties emerged.

### *Day 1*

Bill introduced Ann to Over & Back, explaining that it was a simulated race between Turtle and Rabbit. He demonstrated the features of the program, allowing Ann to see how it worked. The introduction went smoothly with no apparent difficulty for either Bill or Ann. Before starting the tasks of Activity 1 (Table 1), Bill asked Ann to explain what it meant to her to travel at 45 miles per hour. Ann's response was "It means, like … it means, like, if you're going forty-five miles per hour that means that if you time yourself at an hour you would have gone forty-five miles from where you started." Bill probed further by asking Ann what it meant if she traveled for only a half hour. To this Ann responded "you would only go half of forty five." Bill then asked Ann to suppose Rabbit was traveling at 30 feet per second, to which Ann interjected, "So he wouldn't go that far, right?" Bill ignored Ann's comment, instead asking her to say how far Rabbit would go in 2, 3, and 4 seconds. Bill, satisfied with Ann's responses, then moved on to the tasks of Activity 1.

Each task in Activity 1 required Ann to determine the time it would take either Turtle or Rabbit to complete the race when given specific speeds. The last two tasks asked for a winner between Turtle and Rabbit when given their speeds. Ann approached each task in Activity 1 the same way—by determining how many speed-lengths were in the total distance. Presented with the first task of determining Turtle's time over and back when it was assigned a speed of 20 ft/sec over and back, Ann's initial reaction was to count the tick marks that appeared on the segment which represented the distance of 100 feet. Concluding that each interval represented a length of 10 feet, she first counted intervals in pairs and then computed 100÷ 20. In every task thereafter, Ann determined Rabbit's time by dividing the value of the given speed into 100 feet and then doubling the result. At times she would forget that she was to find the time for the total trip, over and back, but recovered easily with prompts from Bill.

The interaction during this first session seemed driven by the tasks in the activity sheet. Bill's agenda was to move Ann through the tasks. He seemed intent in the immediate goal of finding the missing values and completing the activity sheet. After Ann calculated a value Bill would ask her to test her result by running the program. He would then move to another item without asking Ann to explain what she had done, why she had done it, or what she had obtained as a result of what she had done. The conversations were never about the metaphor of Over & Back or about the idea of speed. Ann was succeeding at the tasks.

Ann first experienced visible difficulty with a task in Activity 2, where she was to give Rabbit a speed so that he would travel over and back in six seconds. Her attempts to determine a speed revealed clearly, to us, that Ann's use of division in tasks from Activity 1 was her way to obtain an answer to what she had construed as a measuring task, namely determining how many "speed-lengths" would "fit" in the specified distance.

Excerpt 1 illustrates Ann's use of a "fitting" strategy to find the speed for a specified time. It follows her solutions to the first two tasks of Activity 2 whereby she merely recalled time-speed pairs that she had determined during Activity 1.

*Excerpt 1*

1. Bill: Six is next (*meaning give Rabbit a speed that would make him go over and back in 6 seconds*)
2. Ann: We didn't do that one.
3. Bill: Huh?
4. Ann: We haven't done that one (*meaning that six seconds was not a previous result from Activity 1*).
5. Bill: Uh uh (no).
6. Ann: Notes don't help (*chuckles, rearranging her scratch paper.*)
7. Bill: But that's all right. There's plenty of paper there. We've got plenty of space to write.
8. Ann: Okay. Six … (*writes 100÷15 in long division form, then uses the calculator to find the result*)[2]
9. Bill: So, what did you do, divided fifteen into a hundred?
10. Ann: (*Looks quizzically at the result*). That's wrong. (*Responding to Bill's question:*) Yeah.
11. Bill: That was for a guess-and-test kind of thing?
12. Ann: Yeah. (*Writes "6.6"*).
13. Bill: Okay. And what did you come up with for that?
14. Ann: Six point six, that's over already, and that's for just one way.

Following this, Ann divided 100 by 10, then 100 by 5. Neither Bill nor Ann clarified what quantities Ann was trying to evaluate. Ann, at times, became confused as to whether the divisor was a value for speed or for time. Bill reminded Ann that she already knew how long it would take Rabbit when its speed was 5 ft/sec. Ann responded that the time will be "well over six seconds." Bill then oriented Ann toward thinking about what she had done earlier in Activity 1, in an apparent effort to get her to realize that division was the appropriate operation. We relate this episode in Excerpt 2.

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[2] We use an ellipsis (…) within excerpts to indicate a pause. It does not indicate omitted text.

*Excerpt 2*

1. Bill: But the way you were doing it before (*gestures to scratch paper*), you know, the first, the first things we were working on here. How did you figure out how long it would take him?
2. Ann: We didn't do it … last time. (*meaning 6 seconds*)
3. Bill: I know we didn't do it with five seconds, but how did you figure out how long it would take him if he has a set speed (*points to Rabbit-speed Box*), in this case of five feet per second?
4. Ann: I … added. I think. I don't know.
5. Bill: On all the first problems you were doing, all these (*points to first piece of scratch paper having Ann's calculations*)?
6. Ann: No, I divided.
7. Bill: Yeah.
8. Ann: For the (*inaudible*), for the time ones, to find out the time.
9. Bill: Okay. So if I divide the … What is this time (*points to the Rabbit-speed Box*) going to come up with? Can you tell?
10. Ann: Umm (*looking to the computer screen*) About forty seconds, probably.
11. Bill: Yep (*nods*) Does that sound right to you?
12. Ann: No (*chuckles*). That's way too far.
13. Bill: True. (*Pause*) So, what is it … (*Ann plays with the mouse*) Let's go back and review again. What is the target time we're aiming for here?

Bill led Ann through repeated uses of guess-and-test, prompting Ann to recall the times obtained from previously-tested speeds of 5, 10, 15, and 20 ft/sec, and asking her to observe that those times successively approximated the six seconds. After observing that a speed of 20 ft/sec would result in 10 seconds, Ann gave Rabbit a speed of 30 ft/sec and waited for the computer to give her the time. Remarking that this speed was too slow, she made subsequent adjustments eventually settling on a speed of 33.1 ft/sec—all the while guessing and testing by means of the program.

Bill mentioned, with an evident sense of relief, that they were almost out of time. He asked Ann if there was anything else she would like to try on the computer. Ann wished to test a speed of 0.1 ft/sec to see how long it would take Rabbit to go over and back. Bill said it would take a long, long time and that the bell would ring before Rabbit could finish the race. He asked Ann how she would go about determining how long it would take. Ann responded, "It would be over forty seconds!" Bill pressed Ann to explain how she might determine the time without using the computer, but Ann only gave qualitative estimates based on the screen as she watched Rabbit run. Ann was unable to explain how she would determine the time in this case, despite her previous successes in determining time when she was given an animal's speed.

*Discussion of Day 1*

Earlier we mentioned that Ann's use of division to calculate an amount of time for a specific speed was not grounded in a conceptualization of a multiplicative relationship among speed, distance, and time. Instead, her use of division was tied to her conceptualization of the situations as measurement tasks—ones in which she used the given speed as a measuring stick (speed-length). Her task then was to determine how many speed-lengths would fit in the specified distance. This interpretation explains Ann's actions in dealing with tasks of Activity 2. In these tasks, and despite Bill's suggestions, Ann did not divide distance by time. From her perspective, without knowing the speed-length beforehand she could not possibly measure, for it made no sense to measure without knowing the length of the measuring stick. Thus, there was nothing by which to divide. This explains why Ann repeatedly tested different speed-lengths, but yet did not “see” that dividing the specified time into the distance would give her the speed. To see this would have required Ann to conceptualize the situation differently than as a measuring task. Ann’s conceptualization of the task as a measuring task also explains her inability to determine Rabbit’s time when its speed was 0.1 ft/sec. From a very practical, grounded image of actual “measurings”—from which Ann seemed to be operating—it is quite impractical to measure 100 feet with so small a unit.

Bill ran into difficulty because, rather than help Ann conceptualize the direct proportional relation between completed distance and completed time, he encouraged her use of a guess-and-test, fitting strategy—a strategy that fit Ann’s additive reasoning. She was imagining time as accruing as a result of an additive accrual of speed-lengths over a given distance. Thus, Bill reinforced the very conceptualization that caused Ann to have difficulty.

Excerpt 2 contains a significant event which suggests that miscommunication between Bill and Ann began early and continued throughout the session. To Bill, “the way you were doing it before” (Excerpt 2, ¶ 1) meant that she had divided. To Ann, the way she had been “doing it before” was that she measured distance in units of speed-length, whence “I added” (Excerpt 2, ¶ 4). That is, Bill was oriented to the calculations Ann employed whereas Ann was oriented to how she was thinking about the situation. Her decision to employ division seems to have been made because she had already established an association between a situation of “how many of these are in those” and division as a calculational operation. So, while Bill spoke of calculations, Ann was thinking of measurements; while Ann spoke of calculations as an affiliated operation to a way of thinking of a situation, Bill thought of evaluating one of three quantities in a multiplicative relationship when values of two are already known.

*Day 2*

In the evening after Day 1’s session, Bill and Pat discussed Ann’s progress that day. Bill thought that Ann had done fairly well, but was puzzled as to why she could not answer questions from Activity 2 when she had done so well on Activity 1. Pat made two suggestions: Bill should focus on having Ann make her reasoning more explicit and should work to orient Ann toward thinking about the covariation of distance and time. Pat pointed out to Bill that there was no indication that Ann had made time an explicit quantity in the situations she addressed, and that there was every indication that Ann was thinking of speed as a distance. Pat also reminded Bill of the session they had held prior to the teaching experiment in which Pat focused on proportional correspondence between accumulations of distance and accumulations of time.

Day 2’s session began with Bill asking Ann to recall a problem from the previous day—to determine Turtle’s total time with speeds of 40 ft/sec (over) and 30 ft/sec (back). Bill focused initially on Ann’s reasoning and her understanding of the quantities (Excerpt 3), but his emphasis quickly resembled that of the previous day (Excerpt 4). Excerpt 3 presents their discussion as Bill asked Ann to explain her reasoning.

*Excerpt 3*

1. Bill: Could you go through that once more and show me how you figured out his time (*puts 40 in for the Turtle-Over Box*). I just want to make sure that I understand that you understand. Okay, let’s get this one

out of here. Put a zero (*types 0 for the Rabbit-speed Box*).
2. Ann: Okay.
3. Bill: You can use the paper there (*points to a pile of scratch paper*) and a pencil.
4. Ann: I would … I would take forty, right?
5. Bill: Uh huh.
6. Ann: (*Uses the mouse as an on-screen pointer*) I've got forty there (*Turtle-Over Box*) and I've got thirty here (*Turtle-Back Box*), so I would … I would, um, divide forty into a hundred … which would come up with … eighty … er, it could go in twice.
7. Bill: Uh huh.
8. Ann: And then,
9. Bill: What is that, the twice (*holds up two fingers*)? What is that? What does each of those twices represent, I should say (*holds up two fingers again and shakes them*)?
10. Ann: Forty. They each represent forty.
11. Bill: In distance, they represent that. But what you said, "Goes in there two times." What are those two (*taps desk twice*)? They're not feet are they?
12. Ann: No, they're … forty feet.
13. Bill: Okay. The forty goes into a hundred … two times (*moves finger over on desk*).
14. Ann: (*Begins to fidget*) Yeah.
15. Bill: What does that number two (*holds up two fingers*) … Let's just stop right there and figure it out.
16. Ann: Eighty feet.
17. Bill: No.
18. Ann: Eighty feet.
19. Bill: No, that's the distance.
20. Ann: Yeah! (*Smiles*).
21. Bill: But the two (*holds out two fingers again*) represents something else.
22. Ann: The two … I don't know.
23. Bill: What happened when you went that first forty feet.
24. Ann: … It was one second.
25. Bill: Okay, what happened when you went the second forty feet?
26. Ann: Two seconds.
27. Bill: Okay. When you divided the forty into the hundred and you say it goes in there two times plus some, what else do those two represent (*holds up two fingers*)?
28. Ann: Seconds?
29. Bill: Don't they?
30. Ann: (*Pause*) Yeah.

As seen in Excerpt 3, Bill began the session with a clear conceptual orientation. He asked Ann what quantity each number evaluated (¶s 9 ff.), what was happening in the situation as Turtle's distance increased (¶s 23-26), and what quantity ended up being evaluated when Ann divided 100 by 40 (¶27). He did not explicitly address Ann's basic conception of speed as being a distance, which was evident in her remarks throughout this episode, but he did bring out the covariation of distance and time (¶s 23-26).

Immediately following the episode in Excerpt 3, Bill changed his orientation to focus on the procedure by which to calculate 100 ÷ 40 and to calculate 100 ÷ 30. We relate this next episode in Excerpt 4.

*Excerpt 4*

1. Bill: Let's go through the example.[Portion of transcript omitted.] Tell me what your calculations are on that one and we'll discuss it some.
2. Ann: Okay, forty went into a hundred, right? It goes in two seconds. That would be eighty. (*Pause*) And that leaves twenty left over.
3. Bill: Uh huh.
(*1.5 minutes of procedural discussions are omitted*)
4. Bill: Okay, the reason I asked that now is that you were telling me this two and a half (*points to Ann's written answer, 2.5*) tells me how many forties will go into one hundred, right? (*Ann nods.*) So, you were saying originally, well this is two and a half forties (*points to Ann's scratchwork*) which means two times forty is eighty and a half of a forty is twenty, so that makes the one hundred. But doesn't … does this also tell you the amount of time?
5. Ann: Yeah.
6. Bill: So, there is a direct connection, you see, between that number (*points to 40*) and that (*points to 2.5*).
7. Ann: That (*points to her scratch work*) only tells the amount of time going over.
(*Bill and Ann continue for another 2 minutes to calculate 100 ÷ 30 using long division. Ann then adds the two numbers and tests her answer of 8.8 sec, running Turtle with speeds of 40 ft/sec over and 30 ft/sec back.*)

The episode in Excerpt 3 lasted approximately two minutes; the episode in Excerpt 4 lasted approximately four minutes. Bill spent two minutes on conceptual aspects of speed, distance, and time and four minutes on Ann's doing long division. His one nod to conceptual aspects of the task in Excerpt 4 was to note that "100 ÷ 40 = 2.5" says that there are two and a half 40's in 100, that "100 ÷ 40 = 2.5" also says something about an amount of time, and that there is a direct connection between 40 and 2.5 (Excerpt 4, ¶s 4-6).

Following the exchange in Excerpt 4 Bill asked Ann to explain as best she could the relationship between speed, distance, and time. Ann talked about the previous task, explaining what it meant for Turtle to go over at 40 ft/sec and back at 30 ft/sec. During her explanation Ann made the remark that Turtle would go "ten feet slower" on the way back than on the way over. This caught Bill's attention, and he asked Ann to explain what she meant by "ten feet slower." Ann replied with an example, that if two people were racing at 40 mi/hr and 30 mi/hr, the slower person would always be 10 miles behind the other. Excerpt 5 enters the discussion at this point.

*Excerpt 5*

1. Bill: Well lets say it's a … 24 hour race, okay? We're gonna race all day and all night. And you're going 40 miles per hour (*points to Ann*). I'm going (*gestures to self*) 30 miles per hour. At the end of one hour (*gestures with hand to indicate the passage of time*), how far behind you will I be (*makes a space with fingers on desk for distance*)?
2. Ann: Ummm … 10 feet.
3. Bill: We're racing …
4. Ann: Ten miles.
5. Bill: (*Nods*) Ten miles, okay? Now we keep on racing. We don't stop. We just keep on going. Another hour goes by. How far will I be behind you?
6. Ann: Twenty miles?
7. Bill: Yeah (*nods*).

8. Ann: So, they would just add up?
9. Bill: Sure. Because every hour that we race, you're going 10 miles per hour faster than I am. So I get 10 miles further behind every time we go for another hour … Okay?
10. Ann: (*Nods*) Okay.
11. Bill: Now. Let's try going back to this again (*points to computer*). I don't … I'm not too concerned about which way we go, but let's say … let's just use the rabbit now because he's going to go over and back and we don't have to set two different speeds here. Let's say we want the rabbit to go over and back in … 5 seconds.

Excerpt 5 shows Bill reacting in a very natural, image-building and potentially productive way to Ann's conception of a difference in speeds (¶s 1-9). However, his eagerness to move on to tasks of Activity 2 (¶ 11) suggests that he treated her conception as if it were something to be fixed incidentally instead of as something that indicated a serious problem.

At the end of Excerpt 5 Bill asked for the speed at which Rabbit must travel in order to complete the race in 5 seconds. Ann readily responded 40 ft/sec, "because we did that yesterday." Bill decided to skip several tasks on the sheet and asked Ann for the speed that would give Rabbit a total time of 7.5 seconds so that Ann could not use basic multiplication facts or results from previous activities. Ann began to guess-and-test once more, but Bill interrupted her. Excerpt 6, which begins with Bill's interruption, shows Bill's attempt to help Ann see a direct proportional relation between completed distance and completed time—a suggestion Pat had made the previous evening. The suggestion had been that Bill use line segments for distance and time as a way of representing the two quantities, and that he use the two to help Ann "see" the covariation of distance and time. Bill could then focus on the covariation of distance and time as suggesting a proportional correspondence between partitions of completed distance and completed time.

*Excerpt 6*

1. Bill: Let me just draw something right here (*draws a line segment on paper*). We're going to say that this is the 100 feet that's up there (*draws a tick mark at both ends of the segment; labels them "0" and "100"*), okay? And I'm not for the moment going to divide that up into any distance per se, but we'll just say this is 0 and this is 100. If we have the turtle or the rabbit running at … um … let's say 40 feet per second (*Ann nods*). Down here we're going to have a graph of time (*draws a time line under the distance line*). Okay? (*Ann nods*). This is 0 seconds (*draws a tick mark on the time line's left end*) and this (*draws tick mark on the time line's right end*) is whatever time it takes him to get down to the end (*points to "100" on the distance line*). If he's running … let's say he is running at 40 feet per second (*writes "40 ft/sec"*), can you diagram on there (*points to the distance line*) where he's going to be at each second (*makes a space between his fingers on the distance line*) and where those seconds are on this graph (*points to the time line*) at the same time? Let me just show you what I mean. This ending point here (*highlights the right tick mark on the distance line*) is the same as the ending point here (*highlights the right tick mark on the time line*). So when he reaches from here to the end (*moves pen from left to right on distance line*), he's gone from zero time to whatever that time is at the end (*moves pen from left to right on the time line*).

2. Ann: Is he going back (*indicates over and back*)?
3. Bill: No, let's just take him one way for the moment.
4. Ann: Just one way?
5. Bill: Uh huh.
6. Ann: Well, after 40, if he went 40 feet for one second … he would be … here (*highlights approximately one-third of the distance line*)
7. Bill: Okay.
8. Ann: … after 1 second (*writes "1 sec"*). That would be like here, right?(*marks approximately one-tenth of time line*)
9. Bill: That's what I want to see. Just go ahead and do it, okay? (*Ann extends highlighted segment to about three-fourths of the distance line and writes "2 sec" alongside it. She draws a second tick mark on the time line and labels it "2", marking approximately one-fifth of the total time line. Ann then acknowledges that the 20 ft "left over" on the distance line and marks it with a bracket.*)
11. Bill: Okay. And how long is it going to take him to do that?
12. Ann: It would take him half as much time as it takes this (*taps the second 40 ft*). So it would be half a second.
13. Bill: Okay (*nods*). And where would that be down there? (*gestures to the time line*)
14. Ann: That would be like … over here (*writes "1/2" over a tick mark slightly to the right of the 2 on the time line*). So it would take him . . .
15. Bill: (*Interrupting*) Remember what I was saying on this diagram down here (*time line*) that we want. This is the starting point (*points to the 0 on the time line*). That's the ending point (*points to right end point*).
16. Ann: Ohhh (*draws a new time line below the old. Puts a 0 on the left and a 2 1/2 on the right. Then draws a tick mark about a third of the way from the left and labels it "1", a tick mark about two thirds of the way from the left and labels it "2" and a smaller tick mark about a fifth of the way from the right and labels it "1/2"*).
17. Bill: Okay. Is this the two and a half mark (*points "1/2"*), or is that the two and a half mark (*points "2 1/2," written near the end of the distance line*)?
18. Ann: This one (*writes "1/2" over 2 1/2 at the end of the distance line and scribbles out previous 1/2*).
19. Bill: Okay. Good! Now, let's assume he's going to run at … um, some different speed. Why don't you do the same kind of thing on your own over there (*gestures to scratch paper*). What if he's going to run at, ummm, 45.

Excerpt 6 shows Bill working against himself. For Ann to obviate her need to guess-and-test various speeds, she needed to realize a proportionality between accumulated distance and accumulated time in relation to total distance and total time. But Bill did not orient Ann toward issues of proportionality. Instead, he joined Ann in her world of additive accumulation of distance.

We see two other significant aspects of Excerpt 6. First, the majority of Ann's reasoning was directed toward the distance line alone (¶s 6-12), so that accumulated time remained implicit in her representation of accumulated distance. Second, her markers on the time line, which indicated numbers of seconds passed as Rabbit or Turtle went over (one way), were not placed proportionately with her indications of cumulative distance (¶s 13-18), a feature to which Bill evidently attributed little significance—he was satisfied that Ann determined that the amount of time would be 2.5 seconds. In responding to Bill's request to "do the same kind of thing … if he's going to run at 45" (Excerpt 6, ¶ 19), Ann again focused on the distance line and placed markers on the time line without regard to proportionality between the distance and time partitions. This time, Bill noticed (Excerpt 7).

*Excerpt 7*

1. Bill: Okay, so it's a little less than a quarter. We won't dwell on that for the moment. But here is what I want to come back to. Do you see the distance you have here from zero to one second and one to two seconds (*drags finger along the time line*)?
2. Ann: Uh huh.
3. Bill: Should those be the same distances?
4. Ann: No.
5. Bill: Isn't one second as long as the first and second second?
6. Ann: Yeah. But it can't be perfect.
7. Bill: I'm not saying about your drawing, I'm just saying in reality, though.
8. Ann: Yeah.
9. Bill: If we were graphing time, would we have equal … ?
10. Ann: (*Interrupting*) If you were like a scientist or something and you were graphing time, you would make sure that they were like even. You'd probably have a ruler or something (*pretends to measure the time line with a ruler*).
11. Bill: Hmm. Okay (*nods*). I want to make sure you understand, though, that one second is the same length as the next second and the next second.

Excerpt 7 is significant in that Ann sees "equal intervals" only as something to be accomplished with accurate measuring of an actual drawing (¶s 6-10), whereas Bill saw "equal intervals" as a matter of logical necessity. It is also significant that Bill appeared to accept Ann's statement that the segments would have to be the same size "if you were a scientist or something" and demanded accurate drawings, but he accepted it, evidently, with the significance of his understanding of logical necessity and not with Ann's significance of accurate measuring. We note also that the criterion Bill raised—each second-segment should be the same length—did not raise the issue of lack of proportionality between Ann's segmentations of distance and time.

Bill eventually returned to the original task of determining Rabbit's speed for a time of 7.5 seconds over and back, which he subsequently changed to 7 seconds. In this discussion Ann multiplied 100 feet by 7 seconds and then tried to determine what she should subtract. We have no idea what Ann had in mind with this calculation, and Bill did not ask her. Instead, he understood her to be thinking that Rabbit was traveling at 100 ft/sec, asking her, "That's not correct, right?" Ann replied, "Yeah, that's why you have to subtract it (700) by something." Bill did not pursue Ann's comment, instead he redirected the conversation to the choice of an appropriate calculation (Excerpt 8).

*Excerpt 8*

1. Bill: Ah. Okay. Let me back you up a minute. When you said here a minute ago (*touches the 0 and the 7 on the time line*) that if we're going to go for 7 seconds down to the end half of that 7 seconds, how far will he have gone in this 100 feet (*touches the 0 and the 100 on the distance line*)?
2. Ann: … Half way.
3. Bill: Put your mark down there (*gestures to the two lines; Ann marks the*

*halfway points of both*). How long will it take him to get half way?
4. Ann: (*Pause*) Three and a half seconds.
5. Bill: Uh huh (yes). That's right. So from that is there a way that we can use this (*puts fingers at ends of time line*) to determine the speed (*indicates a part of distance line*) it's going to require him to get up there (*drags a finger along the distance line*)?
6. Ann: (*Softly*) No.
7. Bill: Remember the first thing we were working on this morning?
8. Ann: Uh huh (yes).
9. Bill: He was running at 40 feet per second. What did you do with the 40 feet per second to get the time it took him to go the full length … (*gestures across distance line*) or the 100 feet (*gesturing across distance line again*)?
10. Ann: … I divided it.
11. Bill: (*Nods*) Uh huh (yes). Why?
12. Ann: So I could get the answer.
13. Bill: Okay, but when you divided the 40 into the 100 feet, in effect you were saying that gives me a 40 foot section here (*points to a first part of the distance line*), a 40 foot section here (*points further to the right on the distance line*), and I had 20 left over (*points to a portion at the end of the distance line*), right? (*Ann nods*) So that gave me a second, a second, and a half second.
14. Ann: Uh huh (yes).
15. Bill: What would prevent you from doing the same thing with the seconds?
16. Ann: (*Pauses while looking at her scratch paper*) I don't quite understand (*shakes head*).

Hidden in Excerpt 8 are a number of images and motivations. Pat had explained to Bill that the distance and time segments could be used to show that if total distance were partitioned, then total time would be partitioned proportionately, and that if total time were partitioned, then total distance would be partitioned proportionately. So, if distance were partitioned into 2.5 units, then time would be partitioned into 2.5 units. If time were partitioned into 7 units, then distance would be partitioned into 7 units. In Excerpt 8 it appears that Bill only began speaking of partitions of time as corresponding to partitions of distance in the hope that Ann would make the generalization herself (¶s 1-6), but Ann could not do so (¶ 6). He then attempted to orient Ann toward thinking of a partition of distance into 2.5 units producing a partition of time into 2.5 units, where the unit of time is one second (¶s 7-15). Bill's wording (¶ 13), however, allowed Ann to understand him as talking about measuring distance in units of speed length so as to *produce* a total time. It seems reasonable to conclude that Ann understood "doing the same thing with seconds" (¶ 15) as a request to measure the number of seconds in units of 40 ft in the same way that Bill spoke of measuring total distance in units of 40 ft. The exchange continued with Bill again trying to help Ann see proportionality between distance and time.

*Excerpt 9*

1. Bill: Okay. Now, instead of knowing the speed (*holds thumb and index finger apart*), we know the time. I'm going to travel from here to there (*moves hand from 0 to 100 on Ann's distance line*) in 7 seconds. Okay?
2. Ann: Okay.
3. Bill: If I do that, how far … can you show me on here (*points to time line*), kind of generally speaking … if I do it in seven seconds, how far … you've marked here how far I've traveled in three and a half seconds (*points to the midpoint of both lines*), how far on that

thing (*points to distance line*) would I have traveled in one second? Or two seconds? Or seven seconds?

4. Ann: (*Looks down at the paper. Sounding discouraged.*) The whole thing?
5. Bill: Seven seconds would be the whole thing, okay?(*points to the 7 on the time line* ) How about one second, two seconds? Can you just make marks on there like you're going to put this into the sections showing how far you would go each second (*uses thumb and index finger to indicate succesive intervals*)? (*Ann puts five tick marks on the time line, haphazardly dividing it into seven intervals* ) Okay. And you have (*counts the intervals*) 1-2-3-4-5-6-7 sections. Right? Do those correspond to sections up here (*indicates sections on Ann's distance line*)?
6. Ann: Yeah. (*Pause*) Yeah (*nods*).
7. Bill: (*Nods*) Okay, they do. Now the main thing we're trying to figure out is … what is that distance that he traveled in this one second (*drags finger over a small area of the distance line*). How can I determine that from what you know now?
8. Ann: (*Looks at the paper for a long time*) I'm not sure.

Excerpt 9 shows Bill again trying to orient Ann toward looking at proportional partitions of the two quantities (¶s 3-5). However, he asked Ann only to think about corresponding segments (¶ 5), which Ann could easily do, but which did not assist her in understanding how a given time and distance would impose a given speed-length as a unit by which to measure distance. In order to take advantage of Bill's remarks, Ann would have had to realize not only that seven seconds corresponds to seven "sections" of distance, but that one second (as one-seventh of the total time) would correspond with a section of distance that is one-seventh of the total distance.

Bill continued to raise the issue of corresponding segmentations, and to get Ann to see that division was an appropriate operation. That Ann became totally confused can be seen in the Excerpt 10.

*Excerpt 10*

1. Bill: (*Pauses.*) How many sections do we have up here (*touching the distance line*)?
2. Ann: Seven.
3. Bill: Are they all the same length?
4. Ann: Yeah.
5. Bill: (*Nods*) Yeah. How long is one of them?
6. Ann: One second long?
7. Bill: One second long, but in feet? (*touches the first interval on the distance line*)
8. Ann: (*Shrugs*) I don't know.

In Excerpt 10 Bill tried to draw an explicit connection between seven sections of time somehow going with seven sections of distance (¶s 1-5), but Ann did not see any connection between the two except that one section of distance went with one section of time (¶ 6). Again, to make the connection Bill hoped, Ann would have needed to understand that one second, as one-seventh of the total time, goes with a segment of distance that is also one-seventh of the total distance.

After the exchange in Excerpt 10 Bill said that they would not consider seven seconds any more, and instead, led her to recall her results for speeds of 20 ft/sec and 25 ft/sec, sketching the respective partitions on distance and time lines. Bill called Ann's attention to their earlier determination that it takes five seconds at a speed of 20 ft/sec to go a distance of 100 ft, and that it takes four seconds at a speed of 25 ft/sec to go 100 ft. Bill eventually focused exclusively on getting Ann to see that she needed to divide. Ann's level of discouragement increased.

*Excerpt 11*

1. Bill: You don't have to come up with the number, but how would you calculate the number? (*Portion of transcript omitted.*) This, we said, was going to take him four seconds, this was five seconds, do you see any relationship between this number (*4*) and this number (*100*) that would lead to that (*25*)? This number (*5*) and this number (*100*) that would lead to that (*20*)?
2. Ann: (*Shrugs, then shakes her head.*) No.
3. Bill: What about these two (*points to 100 and 4*). What if I divide or multiply or add or subtract these two? Do I come up with that in any way?
4. Ann: (*Softly.*) I don't know.
5. Bill: Think about it for a second because that's the key right now. You're right on the verge of knowing the answer. (*Long pause.*) What do you think?
6. Ann: I'm not sure.
7. Bill: Well, tell me if you can see any relationships between these two numbers (*points to 100 and 4*). This is four seconds. That's how long it took him to go 100 feet. Okay? … How can I end up with that (*points to 25*) as a speed?
8. Ann: (*Long pause.*) By subtracting?

Out of desperation, Bill resorted to having Ann think about possible numerical relationships between triplets of numbers, such as (4,25,100) and (5,20,100). Ann did not understand Bill's point; she appeared to have given up all effort to make sense of what Bill was requesting. Ann ended the session with her voice quivering and with tears in her eyes.

Bill phoned us immediately after this session with a sense of urgency in his voice. He acknowledged having run out of ideas to help Ann and asked Pat to work with her the next day. Pat agreed to start out the next day's session and then turn it over to Bill. Pat's session with Ann, and Bill's reflections on it, are the subject of this article's sequel (Thompson & Thompson, in press).

*Discussion of Day 2*

Day 2's session can be characterized as a mismatch between Ann's and Bill's conceptualizations of speed and division. Ann understood speed as being a distance, with time being implicit in that it is *produced* by measuring total distance in units of speed-length, and she understood division as a calculation which is affiliated with determining how many of one quantity are contained in another. Bill had sophisticated and multi-faceted understandings of speed and of division which he had encapsulated under "division."

Bill often drew Ann's attention to what she had done (divide) in order to find time when distance and speed were given—something that from Ann's viewpoint had little or nothing to do with the new tasks of finding speed. We suspect that he did this for two reasons. First, Bill possessed strong, albeit implicit, conceptual connections between the appropriateness of dividing to evaluate speed and the appropriateness of dividing to evaluate time. Bill possessed strong pre-understandings of speed as a proportional correspondence between distance and time and of meanings of division as both partitive and quotative. He also understood that the two meanings of division are related as $a \times b = c \Rightarrow (a = c \div b)$ & $(b = c \div a)$, and he evidently related partitive and quotative division imagistically (but perhaps not consciously) by understanding that measurements of a quantity induce a partition on it and partitions of a quantity induce a measurement of it. To Bill the connection between dividing distance by speed to obtain time and the inverse relation of dividing distance by time to obtain speed was transparent and natural.

Second, Bill's sense of proportionality was so strong that, from his point of view, an example was sufficient to make it obvious. To Ann, who viewed speed as a length and dividing as a way of obtaining the measure of the total distance in units of speed-length, the connections among speed, time, distance, and proportionality were far from transparent.

Bill appeared to lack a clear image of what he wanted Ann to understand. Thus, aside from getting answers to the tasks, Bill appeared to lack a sense of direction for the overall session. He introduced two line segments to represent completed distance and completed time, ostensibly with the intent of getting Ann to realize the proportional relation between the two quantities. However, Bill's discussion started with an image that Ann could assimilate to her measurement concept of speed, and jumped to issues of correspondence without his having raised the issue of proportionality. Bill was satisfied once Ann had completed a task—reflecting on neither her method nor her understanding of what she had done. We suspect this happened because Bill's connections among aspects of speed and among meanings of division were so strong that he *saw* (i.e., imputed) appropriate reasoning any time Ann employed an appropriate calculation.

## CONCLUSION

Early in this paper we mentioned the centrality of classroom discourse and communication to the current reform in mathematics education. We stated that if curricular reform is to happen in classrooms, teachers must teach from the basis of a conceptual curriculum. They must be sensitive to children's thinking during instruction—and shape their instructional actions accordingly. This is a lofty goal, one that places many demands and constraints on teachers. In an attempt to gain insight into the cognitive and attitudinal constraints individual teachers might face when attempting to influence children's thinking, we deliberately set up the one-on-one teaching experiment involving Bill and Ann. We thus reduced the social complexity and demands of whole class interaction and were able to focus on Bill's attempt to influence Ann's thinking so that she would end up constructing a concept of speed as a rate.

The one-on-one setting of the teaching experiment revealed the effect Bill's difficulty speaking conceptually about rates had on Ann's understandings, and the effect Ann's understandings had on the development of Bill's dilemma. It is unlikely that Bill and Ann would have influenced each other as they did had Bill been teaching a whole class, for it is likely that other students would have come to his rescue. In the absence of other students, Bill had to confront his inability to help Ann, and this became a roadblock for him. The one-on-one setting of the teaching experiment thus made it more difficult for Bill to deal with the fact of Ann's difficulties, while it made it easier for us to see the impact on Ann of Bill's calculational orientation—his encapsulation of rich and sophisticated meaning within the language of numbers and operations.

Notice that we think it is Bill, and not Ann, who would be rescued in a whole-class setting. With other students present Bill could have "fished" for someone who could give a correct answer to a question. To Bill, a correct answer would have sanctioned his moving on in the lesson. It is unlikely, however, that Ann's conceptual difficulties would have been helped by hearing another student simply give a correct answer to Bill's questions, although she, like Bill, would have been emotionally relieved.

Bill's difficulty during the teaching experiment's second day is both illustrative and symptomatic of a problem we observed in his teaching as we continued to work with him for two years subsequent to this teaching experiment. During that time Bill frequently impressed us with his good grasp of curricular goals and pedagogical principles. We also observed much progress over time in his knowledge, beliefs, and teaching practice. Yet, he often experienced difficulties similar to the one illustrated in the foregoing excerpts, albeit not as dramatic. Those difficulties seemed to be rooted in his language—"language" in a broad sense, as a representational medium. His use of calculational language during class discussions often interfered with his intention to facilitate students' conceptual grasp of a situation. For example, in discussing the situation of trying to make it to an airport 120 miles away when you have 3.5 hours to get there, Bill said "We multiply 3.5 hours by his speed to go 120 miles," instead of something more situationally-attuned, such as "He's going to go at some constant speed for 3.5 hours, and at the end of 3.5 hours he should have traveled 120 miles."

Beneath Bill's tendency to use a computational language was a consistent difficulty in using everyday language to discuss mathematical ideas. This difficulty appeared to stem from the way he had come to represent these ideas to himself. He had come to use arithmetic representationally—he could read a situation into arithmetic expressions, and he used arithmetic expressions to represent his understanding of a situation. This fits well with our theory of quantitative reasoning (Thompson, 1990b; Thompson, 1993), in which we hypothesized that "good" quantitative reasoners will come to use arithmetic in two ways simultaneously—as a representational system, and as a formulaic system to express an evaluation. What we did not foresee was the shortcoming of this development in regard to teaching. Bill's quantitative conceptualizations appeared to be encapsulated in the language of numbers, operations, and procedures. He thus had no other means outside the language of mathematical symbolism and operations to express his conceptualizations. The language of arithmetic served him well as a personal representational system, or as a system for communicating with other competent quantitative reasoners. Yet, as the excerpts illustrate, that language served him poorly when trying to communicate with children who knew the tokens of his language, but had not constructed the meanings and images that Bill had constructed to go along with them.

Bill's story suggests that teachers need to internalize the subtleties in understanding an idea conceptually in order to shape their actions productively. Bill had been forewarned of Ann's likely difficulties, but did not internalize that advice to a scheme of understandings that gave significance to those scenarios. He did not see in Ann's behavior the difficulties for which he had been advised to watch. In addition, to have been able to help Ann, he needed another language besides the language of arithmetic to express his ideas. He needed an unaffected, non technical language.

Bill's story also suggests a hidden consequence of typical whole-class instruction. The common practice of letting students "off the hook" when their difficulty becomes evident—a practice typically motivated by affective considerations—can result in missed opportunites to make cognitive gains. When students' difficulties become evident, it is important to ensure that their conceptual sources be dealt with substantively until there is evidence of resolution. We realize, on the other hand, that nothing is to be gained by keeping a child "on the hook" if the teacher's attention actually contributes to the child's difficulty.

In the sequel to the present article (Thompson & Thompson, in press) we expand three issues already raised: what it means for teachers to have clear images of understanding an idea conceptually, how those images might be expressed in discourse, and what benefits might accrue to students by addressing the conceptual sources of their difficulties.

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# AUTHOR NOTE

FOOTNOTE